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# Harary index and some Hamiltonian properties of graphs

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**Abstract**

For a nontrivial connected graph $G$, its Harary index $H(G)$ is defined as

$$\sum_{\{u, v\} \subseteq V(G)} \frac{1}{d_G(u, v)},$$

where $d_G(u, v)$ is the distance between vertices $u$ and $v$. Hua and Wang (2013), using Harary index, obtained a sufficient condition for the traceable graphs. In this note, we use Harary index to present sufficient conditions for Hamiltonian and Hamilton-connected graphs.


*Keywords:* Harary index; Hamiltonicity; Hamilton-connectivity

## 1. Introduction

We consider only finite undirected graphs without loops or multiple edges. Notation and terminology not defined here follow that in [1]. For a graph $G = (V, E)$, we use $n$ and $e$ to denote its order $|V|$ and size $|E|$, respectively. For two vertices $u$ and $v$ in a graph $G$, we use $d_G(u, v)$ to denote the distance between them. A cycle $C$ in a graph $G$ is called a Hamiltonian cycle of $G$ if $C$ contains all the vertices of $G$. A graph $G$ is called Hamiltonian if $G$ has a Hamiltonian cycle. A path $P$ in a graph $G$ is called a Hamiltonian path of $G$ if $P$ contains all the vertices of $G$. A graph $G$ is called traceable if $G$ has a Hamiltonian path. A graph $G$ is called Hamilton-connected if for each pair of vertices in $G$ there is a Hamiltonian path between them. If $G$ and $H$ are two vertex-disjoint graphs, we use $G \vee H$ to denote the join of $G$ and $H$. We use $C(n, r)$ to denote the number of $r$-combinations of a set with $n$ elements.

In 1993, Ivanciuc et al. [2] and Plavšić et al. [3] independently introduced the Harary index as a molecular descriptor. For a nontrivial connected graph $G$, its Harary index, denoted $H(G)$, is defined as

$$\sum_{\{u,v\} \subseteq V(G)} \frac{1}{d_G(u, v)}.$$

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In [4], Hua and Wang, using Harary index, obtained the following result for the traceable graphs.

**Theorem 1** *([4]). Let G be a connected graph of order $n \geq 4$. If*

$$H(G) \geq \frac{n^2 - 3n + 5}{2},$$

*then G is traceable, unless $G = K_1 \vee (K_{n-3} \cup 2K_1)$, or $K_2 \vee (3K_1 \cup K_2)$, or $K_4 \vee 6K_1$.*

In this note, using the ideas and techniques in [4], we will prove the following theorems on the Hamiltonian and Hamilton-connected graphs.

**Theorem 2.** *Let G be a connected graph of order $n \geq 3$. If*

$$H(G) \geq \frac{n^2 - 2n + 2}{2},$$

*then G is Hamiltonian, unless $G = K_1 \vee (K_1 \cup K_{n-2})$ or $K_2 \vee (K_2^c \cup K_1)$.*

**Theorem 3.** *Let G be a connected graph of order n. If*

$$H(G) \geq \frac{n^2 - 2n + 3}{2},$$

*then G is Hamilton-connected, unless $G = K_2 \vee (K_1 \cup K_{n-3})$ or $K_3 \vee (3K_1)$.*

**Theorem 4.** *Let $G = (X, Y; E)$, where $X = \{x_1, x_2, \ldots, x_n\}$, $Y = \{y_1, y_2, \ldots, y_n\}$, and $n \geq 2$, be a connected bipartite graph. If*

$$H(G) \geq \frac{9n^2 - 3n - 4}{6},$$

*then G is Hamiltonian, unless $G = P_4$, a path having four vertices and three edges.*

**Theorem 5.** *Let G be a 2-connected graph of order $n \geq 12$. If*

$$H(G) \geq \frac{n^2 - 3n + 7}{2},$$

*then G is Hamiltonian or $G = K_2 \vee ((2K_1) \cup K_{n-4})$.*

**Theorem 6.** *Let G be a 3-connected graph of order $n \geq 18$. If*

$$H(G) \geq \frac{n^2 - 4n + 15}{2},$$

*then G is Hamiltonian or $G = K_3 \vee ((3K_1) \cup K_{n-6})$.*

**Theorem 7.** *Let G be a k-connected graph of order n. If*

$$H(G) \geq \frac{2n(n-1) - (k+1)(n-k-1) + 1}{4},$$

*then G is Hamiltonian.*

## 2. Lemmas

In order to prove the theorems above, we need the following results as our lemmas.

**Lemma 1.** *Let G be a graph of order $n \geq 3$ with degree sequence $d_1 \leq d_2 \leq \cdots \leq d_n$. If*

$$d_k \leq k < \frac{n}{2} \Longrightarrow d_{n-k} \geq n - k,$$

*then G is Hamiltonian.*

**Lemma 2.** *Let $G$ be a graph of order $n \geq 3$ with degree sequence $d_1 \leq d_2 \leq \cdots \leq d_n$. If*

$$2 \leq k \leq \frac{n}{2}, \qquad d_{k-1} \leq k \Longrightarrow d_{n-k} \geq n-k+1,$$

*then $G$ is Hamilton-connected.*

**Lemma 3.** *Let $G = (X, Y; E)$ be a bipartite graph such that $X = \{x_1, x_2, \ldots, x_n\}$, $Y = \{y_1, y_2, \ldots, y_n\}$, $n \geq 2$, and*

$$d(x_1) \leq d(x_2) \leq \cdots \leq d(x_n),$$
$$d(y_1) \leq d(y_2) \leq \cdots \leq d(y_n).$$

*If*

$$d(x_k) \leq k < n \Longrightarrow d(y_{n-k}) \geq n-k+1,$$

*then $G$ is Hamiltonian.*

**Lemma 4** *([5]). Let $G$ be a 2-connected graph of order $n \geq 12$. If $e(G) \geq C(n-2, 2) + 4$, then $G$ is Hamiltonian or $G = K_2 \vee ((2K_1) \cup K_{n-4})$.*

**Lemma 5** *([5]). Let $G$ be a 3-connected graph of order $n \geq 18$. If $e(G) \geq C(n-3, 2) + 9$, then $G$ is Hamiltonian or $G = K_3 \vee ((3K_1) \cup K_{n-6})$.*

**Lemma 6** *([5]). Let $G$ be a $k$-connected graph of order $n$. If $e(G) \geq C(n, 2) - (k+1)(n-k-1)/2 + 1$, then $G$ is Hamiltonian.*

Notice that Lemma 1 is Corollary 3 on Page 209 in [6] or Theorem 4.5 on Page 57 in [1], Lemma 2 is Theorem 12 on Page 218 in [6], Lemma 3 is Corollary 5 on Page 210 in [6], and Lemmas 4–6 can be found in [5]. As in [4], we use $\widehat{D}_G(v)$ to denote

$$\sum_{u \in V(G)} \frac{1}{d_G(u, v)}.$$

Thus

$$\widehat{D}_G(v) \leq d(v) + \frac{1}{2}(n-1-d(v)).$$

Hence

$$\begin{aligned} H(G) &= \frac{1}{2} \sum_{v \in V(G)} \widehat{D}_G(v) \leq \frac{1}{2} \sum_{v \in V(G)} \left( d(v) + \frac{1}{2}(n-1-d(v)) \right) \\ &= \frac{1}{2} \sum_{v \in V(G)} \left( \frac{n-1}{2} + \frac{1}{2} d(v) \right) = \frac{n(n-1)}{4} + \frac{1}{4} \sum_{v \in V(G)} d(v). \end{aligned}$$

The above upper bound for $H(G)$ will be used in our proofs in Section 3.

## 3. Proofs

**Proof of Theorem 2.** Let $G$ be a graph satisfying the conditions in Theorem 2. Suppose that $G$ is not Hamiltonian. Then from Lemma 1, there exists an integer $k < \frac{n}{2}$ such that $d_k \leq k$ and $d_{n-k} \leq n-k-1$. Obviously, $k \geq 1$. Therefore

$$\begin{aligned} H(G) &\leq \frac{n(n-1)}{4} + \frac{1}{4} \sum_{v \in V(G)} d(v) \\ &\leq \frac{n(n-1)}{4} + \frac{1}{4} \left( k^2 + (n-2k)(n-k-1) + k(n-1) \right) \end{aligned}$$

$$= \frac{n(n-1)}{4} + \frac{1}{2} + \frac{(n-1)(n-2)}{4} - \frac{(k-1)(2n-3k-4)}{4}$$
$$= \frac{n^2-2n+2}{2} - \frac{(k-1)(k-2)}{4} - \frac{(k-1)(n-2k-1)}{2}.$$

From $H(G) \geq \frac{n^2-2n+2}{2}$, $k \geq 1$, and $n > 2k$, we have that $H(G) = \frac{n^2-2n+2}{2}$, $k = 1$ or $k = 2$ and $n = 2k+1$, $d_1 = \cdots = d_k = k$, $d_{k+1} = \cdots = d_{n-k} = n-k-1$, and $d_{n-k+1} = \cdots = d_n = n-1$.

If $k = 1$, then $d_1 = 1$, $d_2 = d_3 = \cdots = d_{n-1} = n-2$, and $d_n = n-1$. Thus $G = K_1 \vee (K_1 \cup K_{n-2})$, which is not Hamiltonian.

If $k = 2$ and $n = 2k+1$, we have $n = 5$. Therefore $d_1 = 2$, $d_2 = 2$, $d_3 = 2$, $d_4 = 4$, and $d_5 = 4$. Hence $G = K_2 \vee (K_2^c \cup K_1)$, which is not Hamiltonian.

This completes the proof of Theorem 2. □

**Proof of Theorem 3.** Let $G$ be a graph satisfying the conditions in Theorem 3. Suppose that $G$ is not Hamilton-connected. Then from Lemma 2, there exists an integer $k$ with $2 \leq k \leq \frac{n}{2}$ such that $d_{k-1} \leq k$ and $d_{n-k} \leq n-k$. Therefore

$$H(G) \leq \frac{n(n-1)}{4} + \frac{1}{4}\sum_{v \in V(G)} d(v)$$
$$\leq \frac{n(n-1)}{4} + \frac{1}{4}\left(k(k-1) + (n-2k+1)(n-k) + k(n-1)\right)$$
$$= \frac{n(n-1)}{4} + 1 + \frac{(n-1)(n-2)}{4} - \frac{(k-2)(2n-3k-3)}{4}$$
$$= \frac{n^2-2n+3}{2} - \frac{(k-2)(k-3)}{4} - \frac{(k-2)(n-2k)}{2}.$$

From $H(G) \geq \frac{n^2-2n+3}{2}$, $k \geq 2$, and $n \geq 2k$, we have that $H(G) = \frac{n^2-2n+3}{2}$, $k = 2$ or ($k = 3$ and $n = 2k$), $d_1 = \cdots = d_{k-1} = k$, $d_k = \cdots = d_{n-k} = n-k$, and $d_{n-k+1} = \cdots = d_n = n-1$.

If $k = 2$, then $d_1 = 2$, $d_2 = d_3 = \cdots = d_{n-2} = n-2$, and $d_{n-1} = d_n = n-1$. Thus $G = K_2 \vee (K_1 \cup K_{n-3})$, which is not Hamilton-connected.

If $k = 3$ and $n = 2k$, we have that $n = 6$. Therefore $d_1 = 3$, $d_2 = 3$, $d_3 = 3$, $d_4 = 5$, $d_5 = 5$, and $d_6 = 5$. Hence $G = K_3 \vee (3K_1)$, which is not Hamilton-connected.

This completes the proof of Theorem 3. □

**Proof of Theorem 4.** Let $G$ be a graph satisfying the conditions in Theorem 4. Suppose that $G$ is not Hamiltonian. Then from Lemma 3, there exists an integer $k < n$ such that $d(x_k) \leq k$ and $d(y_{n-k}) \leq n-k$. Next we will find an upper bound for $\widehat{D}_G(x_1)$. Let $N_G(x_1) := \{z_1, z_2, \ldots, z_s\}$ be the neighbor of $x_1$, where $s = d(x_1)$. Then $d_G(x_1, z_i) = 1$ for each $z_i \in N_G(x_1)$, $d_G(x_1, x_i) \geq 2$ for each $x_i$ with $2 \leq i \leq n$, and $d_G(x_1, y_i) \geq 3$ for each $y_i \in Y - N_G(x_1)$. Thus

$$\widehat{D}_G(x_1) \leq d(x_1) + \frac{1}{2}(n-1) + \frac{1}{3}(n - d(x_1)) = \frac{5}{6}n - \frac{1}{2} + \frac{2}{3}d(x_1).$$

Similarly, we have that for each $i$ with $2 \leq i \leq n$ and each $j$ with $1 \leq j \leq n$,

$$\widehat{D}_G(x_i) \leq d(x_i) + \frac{1}{2}(n-1) + \frac{1}{3}(n - d(x_1)) = \frac{5}{6}n - \frac{1}{2} + \frac{2}{3}d(x_i),$$
$$\widehat{D}_G(y_j) \leq d(y_j) + \frac{1}{2}(n-1) + \frac{1}{3}(n - d(y_j)) = \frac{5}{6}n - \frac{1}{2} + \frac{2}{3}d(y_j).$$

Therefore

$$H(G) = \frac{1}{2}\sum_{v \in V(G)} \widehat{D}_G(v) \leq \frac{1}{2}\left(\frac{5}{3}n^2 - n + \frac{2}{3}\sum_{i=1}^{n}(d(x_i) + d(y_i))\right)$$
$$\leq \frac{1}{2}\left(\frac{5}{3}n^2 - n + \frac{2}{3}(k^2 + (n-k)n + (n-k)^2 + kn)\right)$$

$$\leq \frac{1}{2}\left(\frac{5}{3}n^2 - n + \frac{2}{3}((k+(n-k))^2 - 2k(n-k) + n^2)\right)$$
$$\leq \frac{1}{2}\left(\frac{5}{3}n^2 - n + \frac{2}{3}(n^2 - 2*1*1 + n^2)\right) = \frac{9n^2-3n-4}{6}.$$

From $H(G) \geq \frac{9n^2-3n-4}{6}$, $1 \leq k < n$, we have that $k = 1$, $n-k = 1$, $d(x_1) = 1$, $d(x_2) = 2$, $d(y_1) = 2$, and $d(y_2) = 1$. Thus $G = P_4$, which is not Hamiltonian.

This completes the proof of Theorem 4. □

**Proof of Theorem 5.** Let $G$ be a graph satisfying the conditions in Theorem 5. Notice that if $G = K_2 \vee ((2K_1) \cup K_{n-4})$, then $H(G) = \frac{n^2-3n+7}{2}$. Suppose that $G$ is not Hamiltonian and $G$ is not $K_2 \vee ((2K_1) \cup K_{n-4})$. Then from Lemma 4, we have that $e(G) \leq C(n-2,2)+3$. Therefore

$$H(G) \leq \frac{n(n-1)}{4} + \frac{1}{4}\sum_{v\in V(G)} d(v)$$
$$\leq \frac{n(n-1)}{4} + \frac{1}{2}e(G) \leq \frac{n(n-1)+2C(n-2,2)+6}{4} \leq \frac{n^2-3n+6}{2},$$

which is a contradiction.

This completes the proof of Theorem 5. □

**Proof of Theorem 6.** Let $G$ be a graph satisfying the conditions in Theorem 6. Notice that if $G = K_3 \vee ((3K_1) \cup K_{n-6})$, then $H(G) = \frac{n^2-4n+15}{2}$. Suppose that $G$ is not Hamiltonian and $G$ is not $K_3 \vee ((3K_1) \cup K_{n-6})$. Then from Lemma 5, we have that $e(G) \leq C(n-3,2)+8$. Therefore

$$H(G) \leq \frac{n(n-1)}{4} + \frac{1}{4}\sum_{v\in V(G)} d(v)$$
$$\leq \frac{n(n-1)}{4} + \frac{1}{2}e(G) \leq \frac{n(n-1)+2C(n-3,2)+16}{4} \leq \frac{n^2-4n+14}{2},$$

which is a contradiction.

This completes the proof of Theorem 6. □

**Proof of Theorem 7.** Let $G$ be a graph satisfying the conditions in Theorem 7. Suppose that $G$ is not Hamiltonian. Then from Lemma 6, we have that $e(G) \leq C(n,2) - (k+1)(n-k-1)/2$. Therefore

$$H(G) \leq \frac{n(n-1)}{4} + \frac{1}{4}\sum_{v\in V(G)} d(v)$$
$$\leq \frac{n(n-1)}{4} + \frac{1}{2}e(G)$$
$$\leq \frac{n(n-1)+2C(n,2)-(k+1)(n-k-1)}{4}$$
$$\leq \frac{2n(n-1)-(k+1)(n-k-1)}{4},$$

which is a contradiction.

This completes the proof of Theorem 7. □

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